

# Horizon Properties of Einstein-Yang-Mills Black Holes

Burkhard Kleihaus

*Department of Mathematical Physics, University College, Dublin, Belfield, Dublin 4, Ireland*

Jutta Kunz, Abha Sood,\* and Marion Wirschins

*Fachbereich Physik, Universität Oldenburg, D-26111 Oldenburg, Germany*

(Dated: February 7, 2008)

We consider static axially symmetric Einstein-Yang-Mills black holes in the isolated horizon formalism. The mass of these hairy black holes is related to the mass of the corresponding particle-like solutions by the horizon mass. The hairy black holes violate the “quasi-local uniqueness conjecture”, based on the horizon charges.

PACS numbers: 04.20.Jb, 04.40.Nr

## I. INTRODUCTION

In Einstein-Maxwell (EM) theory black holes are completely characterized by their mass, their electric and magnetic charge and their angular momentum, i.e. EM black holes have “no hair” [1]. The unique family of stationary Kerr-Newman black holes includes, beside the stationary charged black holes, the stationary Kerr black holes for vanishing electromagnetic charges, the static Reissner-Nordstrøm (RN) black holes for vanishing angular momentum, and the static Schwarzschild black holes, which carry neither electromagnetic charge nor angular momentum. Notably, the stationary EM black hole solutions are axially symmetric, while the static EM black hole solutions are spherically symmetric [1].

The uniqueness theorem (“no hair” theorem) for black holes in EM theory does not generalize to theories with non-abelian fields coupled to gravity. Non-abelian black holes are not completely determined by their global charges, defined at infinity [2, 3]. Furthermore, non-abelian static black holes need not be spherically symmetric, but might possess axial symmetry [4, 5, 6] or discrete symmetries [7] only, i.e. Israel’s theorem does not generalize to theories with non-abelian fields coupled to gravity either.

In particular,  $SU(2)$  Einstein-Yang-Mills (EYM) theory possesses besides embedded abelian black holes sequences of genuinely non-abelian black holes [2, 4, 8]. These hairy black holes possess non-trivial matter fields outside their regular event horizon. Their global charges alone do not determine the spacetimes uniquely. Instead, they are additionally characterized by two integers, the winding number  $n$  (with respect to the azimuthal angle  $\phi$ ) of the non-abelian gauge fields and the node number  $k$  of the gauge field functions [4].

Beside black hole solutions EYM theory also possesses globally regular solutions [3, 9, 10]. These particle-like solutions are obtained from the hairy black hole solu-

tions in the limit of vanishing horizon size. Therefore the particle-like solutions are also characterized by the winding number  $n$  and the node number  $k$  of the gauge fields.

Recently considerable progress concerning the understanding of the non-abelian black hole solutions has been made in the framework of the newly developed isolated horizon formalism [11, 12, 13].

In the isolated horizon formalism one considers spacetimes with an interior boundary, which satisfy quasi-local boundary conditions that insure that the horizon remains isolated [11, 12, 13]. The boundary conditions imply that quasi-local charges can be defined at the horizon, which remain constant in time. In particular one can define a horizon mass, a horizon electric charge and a horizon magnetic charge [12].

Amazingly, the horizon mass of hairy black holes is related in a simple way to their ADM mass and to the ADM mass of the corresponding particle-like solution [11, 12, 13]. This observation suggests the interpretation of hairy black holes as bound states of a particle-like solution and a Schwarzschild black hole [13]. Most interestingly, a “quasi-local uniqueness conjecture” for black holes has been proposed, stating that static black holes are uniquely determined by their horizon area and their horizon electric and magnetic charges [12].

Here we address these issues, raised from the perspective of the isolated horizon framework, for the static  $SU(2)$  EYM solutions. We focus on static axially symmetric black hole solutions, since static spherically symmetric black hole solutions have been considered before [11, 12, 13]. We verify the mass formula for these static axially symmetric black holes, and consider the bound state interpretation. Our main concern is, however, the “quasi-local uniqueness conjecture”, since, as a first check for this conjecture, Corichi, Nucamendi and Sudarsky [12] have proposed to study colored black holes which are static but not spherically symmetric.

---

\*GKSS, D-21502 Geesthacht, Germany

## II. STATIC AXIALLY SYMMETRIC SOLUTIONS

We consider the SU(2) EYM action

$$S = \int \left( \frac{R}{16\pi G} - \frac{1}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) \right) \sqrt{-g} d^4x \quad (1)$$

with  $F_{\mu\nu} = \partial_\mu A_\nu + \partial_\nu A_\mu - ie[A_\mu, A_\nu]$ , Yang-Mills coupling constant  $e$ , and Newton's constant  $G$ .

The static axially symmetric solutions are obtained in isotropic coordinates with metric [4, 10]

$$ds^2 = -fdt^2 + \frac{m}{f}dr^2 + \frac{mr^2}{f}d\theta^2 + \frac{lr^2 \sin^2 \theta}{f}d\varphi^2, \quad (2)$$

and  $f$ ,  $m$  and  $l$  are only functions of  $r$  and  $\theta$ . For the gauge field a purely magnetic ansatz ( $A_0 = 0$ ) is used [4, 10]

$$\begin{aligned} A_\mu dx^\mu = & \frac{1}{2er} [\tau_r^n (H_1 dr + (1 - H_2) r d\theta) \\ & - n(\tau_r^n H_3 + \tau_\theta^n (1 - H_4)) r \sin \theta d\varphi] . \end{aligned} \quad (3)$$

Here the symbols  $\tau_r^n$ ,  $\tau_\theta^n$  and  $\tau_\varphi^n$  denote the dot products of the cartesian vector of Pauli matrices,  $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ , with the spatial unit vectors

$$\begin{aligned} \vec{e}_r^n &= (\sin \theta \cos n\varphi, \sin \theta \sin n\varphi, \cos \theta) , \\ \vec{e}_\theta^n &= (\cos \theta \cos n\varphi, \cos \theta \sin n\varphi, -\sin \theta) , \\ \vec{e}_\varphi^n &= (-\sin n\varphi, \cos n\varphi, 0) , \end{aligned} \quad (4)$$

respectively. Since the fields wind  $n$  times around, while the azimuthal angle  $\varphi$  covers the full trigonometric circle once, we refer to the integer  $n$  as the winding number of the solutions. For  $n = 1$  and  $H_1 = H_3 = 0$ ,  $H_2 = H_4 = w(r)$ , the spherically symmetric ansatz of ref. [2] is recovered.

To fix the residual gauge degree of freedom [4, 10] we choose the gauge condition

$$r\partial_r H_1 - \partial_\theta H_2 = 0 . \quad (5)$$

We consider static axially symmetric solutions which are asymptotically flat, have a finite mass, and are regular at the origin or possess a regular event horizon. The boundary conditions at infinity and along the axes are the same for the regular and black hole solutions. At infinity ( $r = \infty$ ) the boundary conditions are [4, 10]

$$f = m = l = 1 , \quad H_2 = H_4 = \pm 1 , \quad H_1 = H_3 = 0 , \quad (6)$$

implying that the solutions are magnetically neutral.

Along the  $\theta = 0$ -axis and the  $\theta = \pi/2$ -axis  $H_1 = H_3 = 0$ ; for all other functions the derivatives with respect to  $\theta$  vanish on these axes [4, 10].

For the regular solutions the boundary conditions at the origin ( $r = 0$ ) are

$$\begin{aligned} \partial_r f &= \partial_r m = \partial_r l = 0 , \\ H_2 &= H_4 = 1 , \quad H_1 = H_3 = 0 . \end{aligned} \quad (7)$$

The regular horizon of the static black hole solutions resides at  $r_H$ , where  $g_{tt} = -f = 0$  [4, 10]. The equations of motion impose the boundary conditions at the horizon

$$\begin{aligned} f &= m = l = 0 , \\ \partial_\theta H_1 + r\partial_r H_2 &= 0 , \quad r\partial_r H_3 - H_1 H_4 = 0 , \\ r\partial_r H_4 + H_1(H_3 + \text{ctg}\theta) &= 0 . \end{aligned} \quad (8)$$

For black hole solutions, the gauge condition (5) still allows for non-trivial gauge transformations. The gauge is fixed by the condition [4, 10]

$$\partial_\theta H_1 = 0 . \quad (9)$$

## III. BLACK HOLE PROPERTIES

Let us now discuss the properties of the non-abelian black holes. Since they are static and carry no electromagnetic charge, their only global charge is the mass.

The mass  $M$  of the regular and black hole solutions can be obtained directly from the total energy-momentum "tensor"  $\tau^{\mu\nu}$  of matter and gravitation,  $M = \int \tau^{00} d^3r$  [14].

Changing to dimensionless coordinates,  $\frac{x}{(e/\sqrt{4\pi G})r}$ , the dimensionless mass  $\mu = (e/\sqrt{4\pi G})GM$  of the solutions is given by [4, 10]

$$\mu = \frac{1}{2}x^2 \partial_x f|_\infty . \quad (10)$$

Introducing the dimensionless area parameter  $x_\Delta$  of a black hole with horizon area  $A$ ,

$$x_\Delta = \sqrt{A/4\pi} , \quad (11)$$

we present the mass of the static black hole solutions,  $\mu_{\text{bh}}$ , with winding number  $n$  and node number  $k$  as a function of the area parameter in Fig. 1, for the spherically symmetric solutions with  $(n = 1, k = 1)$ ,  $(n = 1, k = 2)$ , and the axially symmetric solutions with  $(n = 2, k = 1)$ . With increasing horizon size, the mass of the hairy black hole solutions approaches the mass of the Schwarzschild solutions. (Clearly, the non-abelian black holes are not characterized by their mass alone, but need further specifications, such as e.g. their winding and node numbers.)

Let us now address the horizon properties of the non-abelian black holes.

The zeroth law of black hole mechanics also holds for the static axially symmetric black hole solutions [4]. The surface gravity  $\kappa$  [7],

$$\kappa^2 = -(1/4)g^{tt}g^{ij}(\partial_i g_{tt})(\partial_j g_{tt}) , \quad (12)$$

is constant on the horizon.

With  $\partial_x^2 f|_{x_H} = 2f_2$  and  $\partial_x^2 m|_{x_H} = 2m_2$  we obtain for  $\kappa$

$$\kappa = \frac{f_2}{\sqrt{m_2}} . \quad (13)$$

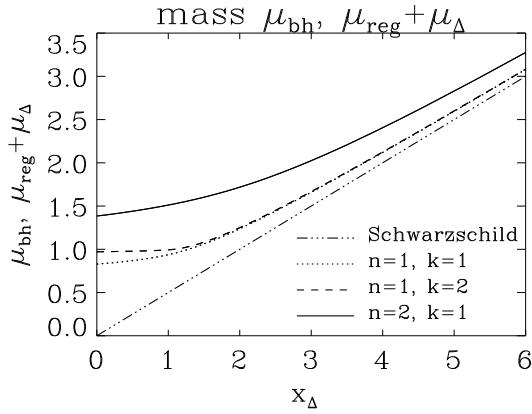


FIG. 1: The dependence of the black hole mass  $\mu_{bh}$  on the area parameter  $x_\Delta$  is shown for the hairy black hole solutions with  $(n = 1, k = 1)$ ,  $(n = 1, k = 2)$ ,  $(n = 2, k = 1)$ . For these solutions the sum  $\mu_{reg} + \mu_\Delta$  is superimposed (and cannot be distinguished graphically). Also shown is the mass of the Schwarzschild solutions.

The isolated horizon framework [11] leads to an intriguing relation between the ADM mass  $\mu_{bh}$  of a black hole with area parameter  $x_\Delta$ , and the ADM mass  $\mu_{reg}$  of the corresponding regular solution [12],

$$\mu_{bh} = \mu_{reg} + \mu_\Delta , \quad (14)$$

where the horizon mass  $\mu_\Delta$  is defined by

$$\mu_\Delta = \int_0^{x_\Delta} \kappa(x'_\Delta) x'_\Delta dx'_\Delta . \quad (15)$$

This mass formula (14) has been shown previously to hold for the numerically obtained static spherically symmetric SU(2) solutions [12]. Here we show, that it also holds for the static axially symmetric solutions. We demonstrate this relation in Fig. 1, where the sum of the mass of the regular solution and the horizon mass,  $\mu_{reg} + \mu_\Delta$ , is superimposed on the black holes mass,  $\mu_{bh}$ , for the EYM solutions with  $(n = 1, k = 1)$ ,  $(n = 1, k = 2)$ , and  $(n = 2, k = 1)$ . The curves agree within the numerical accuracy.

For distorted horizons, relation (14) has been verified previously in Einstein-Yang-Mills-Higgs (EYMH) theory [5, 6], where it generalizes to include bifurcating branches of non-abelian black hole solutions [5, 13].

The isolated horizon formalism also leads to a heuristic physical model of hairy black holes, which suggests to interpret a hairy black hole as a bound state of a regular solution and a Schwarzschild black hole [13]. Rewriting relation (14) for the mass  $\mu_{bh}$  of the black hole solution with horizon radius  $x_\Delta$  by introducing the mass  $\mu_S = x_\Delta/2$  of the Schwarzschild black hole with horizon radius  $x_\Delta$ ,

$$\mu_{bh} = \mu_{reg} + \mu_S + (\mu_\Delta - \mu_S) , \quad (16)$$

yields for the binding energy  $\mu_{bind}$  of the system,

$$\mu_{bind} = \mu_\Delta - \mu_S . \quad (17)$$

In Fig. 2 we show the binding energy  $\mu_{bind}$  for the spherically symmetric solutions with  $(n = 1, k = 1)$ ,  $(n = 1, k = 2)$ , and the axially symmetric solutions with  $(n = 2, k = 1)$ . When the binding energy tends to the negative value of the mass of the regular solution for  $x_\Delta \rightarrow \infty$ , this confirms the mass relation for the regular solution [13],

$$\mu_{reg} = \frac{1}{2} \int_0^\infty (1 - 2\kappa(x'_\Delta) x'_\Delta) dx'_\Delta . \quad (18)$$

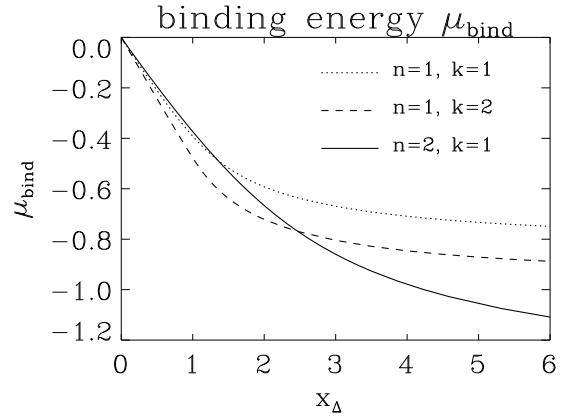


FIG. 2: The dependence of the binding energy  $\mu_{bind}$  on the area parameter  $x_\Delta$  is shown for the hairy black hole solutions with  $(n = 1, k = 1)$ ,  $(n = 1, k = 2)$ ,  $(n = 2, k = 1)$ .

Let us now turn to the “quasi-local uniqueness conjecture” for black holes, put forward in ref. [12], which states that static black holes are uniquely determined by their horizon area and their horizon electric and magnetic charges. Here the non-abelian electric and magnetic charge of the horizon [11, 12] appear as the crucial quantities.

The non-abelian magnetic charge of the horizon is defined via the surface integral over the horizon [11, 12]

$$P_\Delta^{\text{YM}} = \frac{1}{4\pi} \oint \sqrt{\sum_i \left( F_{\theta\varphi}^i \right)^2} d\theta d\varphi , \quad (19)$$

and the non-abelian horizon electric charge is defined analogously with the dual field strength tensor [11, 12].

For the hairy black hole solutions in SU(2) EYM theory, the horizon electric charge is identically zero. Only the horizon magnetic charge is non-trivial and needs to be considered.

For the spherically symmetric solutions, the “quasi-local uniqueness conjecture” holds [12]. For a given value

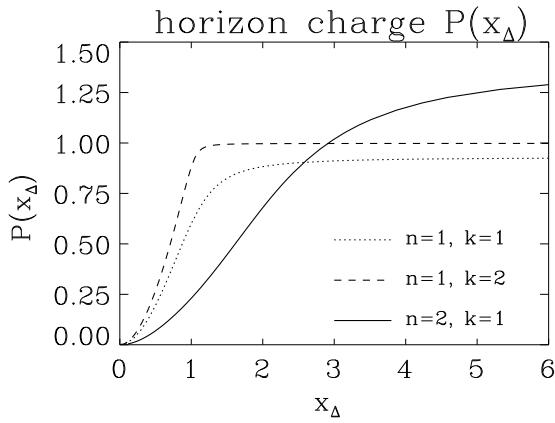


FIG. 3: The dependence of the non-abelian horizon magnetic charge  $P(x_\Delta)$  on the area parameter  $x_\Delta$  is shown for the hairy black hole solutions with  $(n = 1, k = 1)$ ,  $(n = 1, k = 2)$ ,  $(n = 2, k = 1)$ .

of the area parameter, the horizon magnetic charge increases with increasing node number  $k$ , reaching the constant value of the RN solution with unit charge (for horizon  $x_\Delta \geq 1$ ) in the limit  $k \rightarrow \infty$ .

However, the conjecture should also hold when the axially symmetric black hole solutions with winding number  $n > 1$  are taken into account [12]. To address this cru-

cial point, we have reanalyzed the set of hairy black hole solutions with winding number  $n = 2$  and node number  $k = 1$ . We exhibit the horizon magnetic charge  $P_\Delta$  of these non-abelian black hole solutions as a function of the area parameter  $x_\Delta$  in Fig. 3, together with the horizon magnetic charge of the spherically symmetric solutions with one and two nodes.

Whereas the curves of the spherically symmetric solutions do not intersect, the curve of the axially symmetric ( $n = 2, k = 1$ ) solution intersects the curves of both spherically symmetric solutions. In fact, it intersects the curves of all the spherically symmetric solutions, since they increase monotonically with  $k$ , but have  $P_\Delta < 1$ . Furthermore it also intersects the RN curve,  $P_\Delta = 1$  (for  $x_\Delta \geq 1$ ).

Thus when black hole solutions with a distorted horizon are included, the “quasi-local uniqueness conjecture” holds no longer. Therefore the need for a modified uniqueness conjecture persists. The set of two integers  $(n, k)$ , used to characterize the hairy black holes with a given mass or horizon area, apparently cannot be replaced by the single quantity of the horizon magnetic charge. Another horizon property would have to be included in the uniqueness conjecture. The deformation of the horizon could serve to distinguish black holes solutions with equal horizon area and horizon magnetic charge [15].

We will revisit the uniqueness conjecture in connection with rotating EYM black holes [8].

---

[1] W. Israel, Commun. Math. Phys. 8 (1968) 245;  
 D. C. Robinson, Phys. Rev. Lett. 34 (1975) 905;  
 P. Mazur, J. Phys. A 15 (1982) 3173.

[2] M. S. Volkov and D. V. Galt'sov, Sov. J. Nucl. Phys. 51 (1990) 747;  
 P. Bizon, Phys. Rev. Lett. 64 (1990) 2844;  
 H. P. Künzle and A. K. M. Masoud-ul-Alam, J. Math. Phys. 31 (1990) 928.

[3] see e.g. M. S. Volkov and D. V. Gal'tsov, Phys. Rept. 319 (1999) 1.

[4] B. Kleihaus and J. Kunz, Phys. Rev. Lett. 79 (1997) 1595;  
 B. Kleihaus and J. Kunz, Phys. Rev. D57, 6138 (1998).

[5] B. Kleihaus and J. Kunz, Phys. Lett. B494 (2000) 130.

[6] B. Hartmann, B. Kleihaus, and J. Kunz, Phys. Rev. Lett. 86, 1423 (2001);  
 B. Hartmann, B. Kleihaus, and J. Kunz, Axially symmetric monopoles and black holes in Einstein-Yang-Mills-Higgs theory, preprint hep-th/0108129, Phys. Rev. D, in press.

[7] S. A. Ridgway and E. J. Weinberg, Phys. Rev. D52 (1995) 3440.

[8] M. S. Volkov and N. Straumann, Phys. Rev. Lett. 79 (1997) 1428;  
 O. Brodbeck, M. Heusler, N. Straumann and M. S. Volkov, Phys. Rev. Lett. 79 (1997) 4310;  
 B. Kleihaus and J. Kunz, Phys. Rev. Lett. 86 (2001) 3704.

[9] R. Bartnik and J. McKinnon, Phys. Rev. Lett. 61 (1988) 141.

[10] B. Kleihaus and J. Kunz, Phys. Rev. Lett. 78 (1997) 2527;  
 B. Kleihaus and J. Kunz, Phys. Rev. D57, 834 (1998).

[11] A. Ashtekar, C. Beetle, S. Fairhurst, Class. Quant. Grav. 17 (2000) 253;  
 A. Ashtekar, Annalen Phys. 9 (2000) 178;  
 A. Ashtekar, S. Fairhurst, and B. Krishnan, Phys. Rev. D62 (2000) 104025.

[12] A. Corichi, and D. Sudarsky, Phys. Rev. D61 (2000) 101501;  
 A. Corichi, U. Nucamendi, and D. Sudarsky, Phys. Rev. D62 (2000) 044046;  
 A. Corichi, U. Nucamendi, and D. Sudarsky, A mass formula for EYM solitons, preprint gr-qc/0106084, Phys. Rev. D, in press.

[13] A. Ashtekar, A. Corichi and D. Sudarsky, Class. Quantum Grav. 18 (2001) 919.

[14] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972)

[15] Deformation parameters could be the length of the equatorial circumference of the horizon  $L_e$ , the length of the polar circumference  $L_p$ , or their ratio  $L_e/L_p$  [4].